A beginner's guide to analysis on metric spaces

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Abstract

These notes deal with various kinds of distance functions and related properties and measurements for sets and functions.

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1 Basic notions

Let M be a nonempty set. By a *distance function* on M we mean a real-valued function $d(x, y)$ defined for $x, y \in M$ such that (i) $d(x, y) \geq 0$ for all $x, y \in M$, (ii) $d(x, y) = 0$ if and only if $x = y$, and (iii)

$$d(y, x) = d(x, y) \tag{1.1}$$

for all $x, y \in M$. If moreover the triangle inequality holds, which is to say that

$$d(x, z) \leq d(x, y) + d(y, z) \tag{1.2}$$

for all $x, y, z \in M$, then the distance function $d(\cdot, \cdot)$ is said to be a *metric* on M. More generally, if there is a real number $C \geq 1$ such that

$$d(x, z) \leq C(d(x, y) + d(y, z)) \tag{1.3}$$

for all $x, y, z \in M$, then the distance function $d(\cdot, \cdot)$ is said to be a *quasimetric* on M, with constant C.

A basic example occurs with the real numbers $\mathbf{R}$ equipped with the standard distance function. Recall that if x is a real number, then the absolute value of x is denoted $|x|$ and is defined to be equal to x is $x \geq 0$, and to be equal to $-x$ if $x \leq 0$. Thus the absolute value of a real number is always a nonnegative real number, the absolute value of a real number is equal to 0 if and only if the real number itself is equal to 0, and the absolute value of a product of two real numbers is equal to the product of the absolute values of the two real numbers individually. Furthermore,

$$|x + y| \leq |x| + |y| \tag{1.4}$$

for all real numbers x, y, which is also referred to as a triangle inequality. It follows easily that the standard distance function $|x - y|$ on the real numbers is in fact a metric.

A distance function $d(x, y)$ on a nonempty set M is said to be an *ultrametric* if

$$d(x, z) \leq \max(d(x, y), d(y, z)) \tag{1.5}$$

for all $x, y, z \in M$. Thus an ultrametric is automatically a metric. One might consider the weaker condition that there be a real number $C' \geq 1$ such that

$$d(x, z) \leq C' \max(d(x, y), d(y, z)) \tag{1.6}$$

for all $x, y, z \in M$. Of course this would imply that the distance function is a quasimetric, and conversely this condition holds for quasimetrics with C' equal to 2 times the quasimetric constant. In other words, this weaker condition is equivalent to saying that the distance function is a quasimetric, except perhaps with a slightly different constant.

Let q be a positive real number. If $q \geq 1$, then

$$(a + b)^q \leq 2^{q-1} (a^q + b^q) \tag{1.7}$$

for all nonnegative real numbers a, b. This amounts to the convexity of the function t^q on the nonnegative real numbers. It follows that the distance

function $|x - y|^q$ on the real line is a quasimetric with constant 2^{q-1} when $q \geq 1$. More generally, if $d(x, y)$ is a quasimetric on a nonempty set M, then $d(x, y)^q$ is also a quasimetric on M when $q \geq 1$, where the quasimetric constant C for $d(x, y)$ is replaced with $2^{q-1} C^q$.

Now suppose that $0 < q \leq 1$. In this case we have that

$$(a + b)^q \leq a^q + b^q \tag{1.8}$$

for all nonnegative real numbers a, b. Indeed,

$$\begin{aligned} a + b &\leq (a^q + b^q) \max(a^{1-q}, b^{1-q}) \\ &= (a^q + b^q) \max(a^q, b^q)^{(1-q)/q} \\ &\leq (a^q + b^q)\,(a^q + b^q)^{(1-q)/q} \\ &= (a^q + b^q)^{1/q}. \end{aligned} \tag{1.9}$$

Thus we get in this case that the distance function $|x - y|^q$ on the real line is a metric when $0 < q \leq 1$. If $d(x, y)$ is a quasimetric on a nonempty set M with constant C, then $d(x, y)^q$ is a quasimetric on M with constant C^q when $0 < q \leq 1$, and in particular $d(x, y)^q$ is a metric on M if $d(x, y)$ is.

For ultrametrics the situation is much more straightforward. Namely, if $d(x, y)$ is an ultrametric on a nonempty set M and q is a positive real number, then $d(x, y)^q$ is also an ultrametric on M. If more generally $d(x, y)$ is a distance function on M which satisfies (1.6) for some $C' \geq 1$, then $d(x, y)^q$ satisfies (1.6) with C' replaced with $(C')^q$.

Let $d(x, y)$ be a distance function on a nonempty set M. We say that $d(x, y)$ is *almost a metric* with constant $C \geq 1$ if

$$d(w_0, w_l) \leq C \sum_{j=1}^{l} d(w_j, w_{j-1}) \tag{1.10}$$

for any finite sequence of points $w_0, w_1, \ldots, w_l$ in M. If we restrict our attention to the case where $l = 1$, then this is the same as the quasimetric condition. If $d(x, y)$ is a metric, then it is also almost a metric with $C = 1$, by repeating the triangle inequality. However, if q is a positive real number such that $q > 1$, then the distance function $|x - y|^q$ on the real line is a quasimetric but it is not almost a metric, as one can check using simple examples.

Suppose that $d(x, y)$ is a distance function on a nonempty set M which is almost a metric with constant C. Define $\rho(x, y)$ for $x, y \in M$ to be the

infimum of the sums $\sum_{j=1}^{l} d(w_j, w_{j-1})$ over all finite sequences $w_0, \ldots, w_l$ of points in M such that $w_0 = x$ and $w_l = y$. We have that

$$C^{-1}\, d(x, y) \le \rho(x, y) \le d(x, y) \tag{1.11}$$

for all $x, y \in M$, where the first inequality follows from the assumption that $d(x, y)$ is almost a metric with constant C, and the second inequality is automatic from the definition of $\rho(x, y)$. It is easy to see that $\rho(x, y)$ is a distance function on M, and in fact it is a metric, because the triangle inequality for $\rho(x, y)$ is also an automatic consequence of the definition of $\rho(x, y)$.

If $d(x, y)$ is a distance function on a nonempty set M, then we say that $d(x, y)$ is *almost an ultrametric* with constant $C' \ge 1$ if

$$d(w_0, w_l) \le C' \max(d(w_0, w_1), \ldots, d(w_{l-1}, w_l)) \tag{1.12}$$

for any finite sequence of points $w_0, \ldots, w_l$ in M. When $l = 1$ this is the same as (1.6), and an ultrametric is almost an ultrametric with constant equal to 1. On the real line, the distance function $|x-y|^q$ is not almost an ultrametric for any positive real number q, as one can verify.

Let $d(x, y)$ be a distance function on a nonempty set M which is almost an ultrametric with constant C'. Define $\sigma(x, y)$ for $x, y \in M$ to be the infimum of

$$\max(d(w_0, w_1), \ldots, d(w_{l-1}, w_l)) \tag{1.13}$$

over all finite sequences $w_0, \ldots, w_l$ of points in M with $w_0 = x$ and $w_l = y$. Thus

$$(C')^{-1}\, d(x, y) \le \sigma(x, y) \le d(x, y) \tag{1.14}$$

for all $x, y \in M$, where the first inequality follows from the assumption that $d(x, y)$ is almost an ultrametric with constant C', and the second condition is immediate from the definition. Clearly $\sigma(x, y)$ is a distance function, and in fact it is an ultrametric, because it satisfies the ultrametric version of the triangle inequality by construction.

Suppose that M is a nonempty set, E is a nonempty subset of M, and that $d(x, y)$ is a distance function on M. One can restrict $d(x, y)$ to E to get a distance function on E. If $d(x, y)$ is a metric, quasimetric, ultrametric, etc., on M, then the same condition will hold automatically on E as well. However, depending on the structure of E, it may be that the restriction of the distance function to E satisfies conditions that do not work on M.

As a basic example, let M be the real line, and let E be the usual Cantor middle-thirds set, as in [15]. By definition, one starts with the unit interval $[0, 1]$ in the real line, consisting of all real numbers x such that $0 \le x \le 1$, one removes the open middle third of this interval, leaving the two intervals $[0, 1/3]$ and $[2/3, 1]$, one removes the open middle thirds of these two intervals to get four closed intervals of length $1/9$, and so on. At the kth stage of the construction one has a set consisting of 2^k closed intervals of length $1/3^k$, and the union of these intervals can be denoted E_k. By construction, $E_{k+1} \subseteq E_k$ for all k, and the Cantor set E is defined to be the intersection of all of the E_k's. The restriction of the usual distance function $|x - y|$ to the Cantor set is almost an ultrametric.

For both the notions of a distance function $d(x, y)$ being almost a metric or almost an ultrametric, one can consider more general conditions with a lower bound in terms of some function of $d(x, y)$, in place of simply a positive multiple of $d(x, y)$. One could still define $\rho(x, y)$ or $\sigma(x, y)$ as before, with $\rho(x, y), \sigma(x, y) \le d(x, y)$ automatically, and with lower bounds in terms of a function of $d(x, y)$. One would again have that $\rho(x, y)$ is a metric or that $\sigma(x, y)$ is an ultrametric, as appropriate, by construction. This type of condition occurs for instance with more general Cantor sets, in which the sizes of the open intervals being removed are more variable. Of course these sizes are also relevant for the constants involved even if one has lower bounds in terms of multiples of the original distance function.

Remark 1.15 A result of Macías and Segovia [12] states that for each positive real number $C_1 \ge 1$ there are positive real numbers a, C_2 with $a \le 1$ and $C_2 \ge 1$ such that the following is true. Let M be a nonempty set and $d(x, y)$ a distance function on M. If $d(x, y)$ is a quasimetric with constant C_1, then $d(x, y)^a$ is almost a metric with constant C_2.

Let n be a positive integer, and let $\mathbf{R}^n$ denote the usual space of n-tuples of real numbers. Thus $x \in \mathbf{R}^n$ can be written explicitly as $x = (x_1, \ldots, x_n)$, where each component x_j, $1 \le j \le n$, is a real number. If $x, y \in \mathbf{R}^n$, then $x + y$ is defined to be the element of $\mathbf{R}^n$ whose jth component is equal to the sum of the jth components of x, y. If $x \in \mathbf{R}^n$ and t is a real number, then the scalar product $t\,x$ is defined to be the element of $\mathbf{R}^n$ whose jth component is equal to the product of t and the jth component of x. In this manner $\mathbf{R}^n$ is a real vector space of dimension n.

Let $N(x)$ be a real-valued function on $\mathbf{R}^n$ such that $N(x) \ge 0$ for all $x \in \mathbf{R}^n$, $N(x) = 0$ if and only if $x = 0$, and

$$N(t\,x) = |t|\,N(x) \tag{1.16}$$

for all real numbers t and all $x \in \mathbf{R}^n$. If also $N(x)$ satisfies the triangle inequality

$$N(x+y) \le N(x) + N(y) \tag{1.17}$$

for all $x, y \in \mathbf{R}^n$, then N is said to define a *norm* on $\mathbf{R}^n$. In the presence of the other conditions, one can check that the triangle inequality is equivalent to the convexity of the closed unit ball associated to N. In other words, in the presence of the other conditions, N is a norm if and only if for each $x, y \in \mathbf{R}^n$ with $N(x), N(y) \le 1$ and each real number t with $0 \le t \le 1$ we have that $N(t\,x + (1-t)\,y) \le 1$ too. As a generalization of the triangle inequality, if N satisfies the first set of conditions, then we say that N is a *quasinorm* with constant $C \ge 1$ if

$$N(x+y) \le C(N(x) + N(y)) \tag{1.18}$$

for all $x, y \in \mathbf{R}^n$.

As a basic family of examples, let p be a positive real number, and define $\|x\|_p$ for $x \in \mathbf{R}^n$ by

$$\|x\|_p = \Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p}. \tag{1.19}$$

Let us extend this to $p = \infty$ by putting

$$\|x\|_\infty = \max(|x_1|, \ldots, |x_n|). \tag{1.20}$$

These functions clearly satisfy the first set of conditions mentioned in the previous paragraph, namely $\|x\|_p$ is a nonnegative real number which is equal to 0 exactly when $x = 0$, and $\|t\,x\|_p$ is equal to the product of $|t|$ and $\|x\|_p$ for all real numbers t and $x \in \mathbf{R}^n$.

When $p = 1, \infty$ it is easy to see directly from the definition that $\|x\|_p$ satisfies the triangle inequality and is therefore a norm on $\mathbf{R}^n$. This is also true when $1 < p < \infty$, as one can see by checking the convexity of the closed unit ball of $\|x\|_p$. To be more precise, one can reduce this to the convexity of the function u^p on the nonnegative real numbers when $p \ge 1$. When

$0 < p < 1$ and $n \ge 2$, $\|x\|_p$ does not define a norm on $\mathbf{R}^n$. It does define a quasinorm, and indeed one has that

$$\|x + y\|_p^p \le \|x\|_p^p + \|y\|_p^p \tag{1.21}$$

for all $x, y \in \mathbf{R}^n$ when $0 < p < 1$.

If $N(x)$ is a norm on $\mathbf{R}^n$, then $N(x - y)$ defines a metric on $\mathbf{R}^n$. If $N(x)$ is a quasinorm on $\mathbf{R}^n$, then $N(x - y)$ is a quasimetric on $\mathbf{R}^n$. For $\|x\|_p$ with $0 < p < 1$ we have that $\|x - y\|_p^p$ defines a metric on $\mathbf{R}^n$.

For any positive real number p and $x \in \mathbf{R}^n$ we have that

$$\|x\|_\infty \le \|x\|_p. \tag{1.22}$$

Conversely we also have that

$$\|x\|_p \le n^{1/p}\, \|x\|_\infty, \tag{1.23}$$

and in particular $\|x\|_p$ tends to $\|x\|_\infty$ as $p \to \infty$. More generally, if p, q are positive real numbers with $p < q$, then

$$\|x\|_q \le \|x\|_p \tag{1.24}$$

for all $x \in \mathbf{R}^n$, as one can show using the $q = \infty$ case. One also has that

$$\|x\|_p \le n^{(1/p)-(1/q)}\, \|x\|_q, \tag{1.25}$$

which can be verified using the convexity of the function $u^{q/p}$ on the nonnegative real numbers.

2 Lipschitz conditions

Let M be a nonempty set, and let $d(x, y)$ be a real-valued function defined for $x, y \in M$. We say that $d(x, y)$ is a *semidistance function* if $d(x, y) \ge 0$ and $d(y, x) = d(x, y)$ for all $x, y \in M$. If also $d(x, y)$ satisfies the triangle inequality, so that

$$d(x, z) \le d(x, y) + d(y, z) \tag{2.1}$$

for all $x, y, z \in M$, then $d(x, y)$ is said to be a *semimetric*. A real-valued function $N(x)$ on $\mathbf{R}^n$ is called a *seminorm* if $N(x) \ge 0$ for all $x \in \mathbf{R}^n$, $N(t\,x) = |t|\, N(x)$ for all real numbers t and all $x \in \mathbf{R}^n$, and

$$N(x + y) \le N(x) + N(y) \tag{2.2}$$

for all $x, y \in \mathbf{R}^n$. If $N(x)$ is a seminorm on $\mathbf{R}^n$, then $N(x-y)$ is a semimetric on $\mathbf{R}^n$.

Suppose that M is a nonempty set and that $d(x, y)$ is a semidistance function on M. Define $\rho(x, y)$ for $x, y \in M$ to be the infimum of the sums

$$\sum_{j=1}^{l} d(w_j, w_{j-1}) \tag{2.3}$$

over all finite sequences $w_1, \ldots, w_l$ of points in M such that $w_0 = x$ and $w_l = y$. Thus $\rho(x, y)$ is a semidistance function on M such that

$$\rho(x, y) \le d(x, y) \tag{2.4}$$

for all $x, y \in M$, and in fact $\rho(x, y)$ is a semimetric because it satisfies the triangle inequality by construction. Conversely if $\tau(x, y)$ is a semimetric on M such that $\tau(x, y) \le d(x, y)$ for all $x, y \in M$, then $\tau(x, y) \le \rho(x, y)$ for all $x, y \in M$.

Let M_1, M_2 be nonempty sets with semidistance functions d_1, d_2, respectively. If f is a mapping from M_1 to M_2 and L is a nonnegative real number, then we say that f is *L-Lipschitz* with respect to these semidistance functions if

$$d_2(f(x), f(y)) \le L\, d_1(x, y) \tag{2.5}$$

for all $x, y \in M_1$.

Notice that for any mapping $f : M_1 \to M_2$ we have that

$$d_2(f(x), f(y)) \tag{2.6}$$

defines a semidistance function on M_1, since d_2 is a semidistance function on M_2. If d_2 is actually a semimetric on M_2, then (2.6) is a semimetric on M_1. Of course the condition that f be L-Lipschitz is the same as saying that the semidistance function (2.6) is less than or equal to L times $d_1(x, y)$.

Suppose that M_3 is another nonempty set equipped with a semidistance function d_3, and that f_1, f_2 are mappings from M_1 to M_2 and from M_2 to M_3 which are L_1, L_2-Lipschitz for some $L_1, L_2 \ge 0$, respectively. Then the composition $f_2 \circ f_1$, which is the mapping from M_1 to M_3 defined by

$$(f_2 \circ f_1)(x) = f_2(f_1(x)), \tag{2.7}$$

is $L_1\, L_2$ Lipschitz, i.e., one multiplies the Lipschitz constants.

As a special case, let M be a nonemtpy set with a semidistance function $d(x, y)$, and let f be a real-valued function on M. We say that f is L-Lipschitz for some $L \ge 0$ if

$$|f(x) - f(y)| \le L\, d(x, y) \tag{2.8}$$

for all $x, y \in M$. This is the same as saying that f is L-Lipschitz as a mapping from M into the real line equipped with the standard metric $|s-t|$, $s, t \in \mathbf{R}$.

For a real-valued function f on M the property of being L-Lipschitz is equivalent to

$$f(x) \le f(y) + L\, d(x, y) \tag{2.9}$$

for all $x, y \in M$. In particular, if d is a semimetric and p is a point in M, then the function $f_p(x) = d(x, p)$ is automatically 1-Lipschitz.

By contrast, suppose that we take M to be the real line equipped with the quasimetric $|x - y|^q$ where $q > 1$, and that f is a real-valued Lipschitz function with respect to this quasimetric on the domain, which means that

$$|f(x) - f(y)| \le L\, |x - y|^q \tag{2.10}$$

for some $L \ge 0$ and all $x, y \in \mathbf{R}$. In this event one can check that f must be a constant function.

Let M be a nonempty set with semidistance function d, and suppose that $\mathbf{R}^n$ is equipped with a seminorm N. Let f be an L-Lipschitz mapping from M to $\mathbf{R}^n$ with respect to the semimetric associated to this seminorm, which is to say that

$$N(f(x) - f(y)) \le L\, d(x, y) \tag{2.11}$$

for all $x, y \in M$. If t is a real number, then $t\, f(x)$ is $|t|\, L$-Lipschitz as a mapping from M to $\mathbf{R}^n$. If f_1, f_2 are L_1, L_2-Lipschitz mappings from M to $\mathbf{R}^n$, then the sum $f_1 + f_2$ is $(L_1 + L_2)$-Lipschitz.

Now let us specialize to the case of real-valued functions, and the usual absolute values on the real line. Specifically, suppose that f_1, f_2 are real-valued functions on M which are L_1, L_2-Lipschitz, respectively. One can check that the maximum and minimum of f_1, f_2 are $\max(L_1, L_2)$-Lipschitz functions on M. Under the assumption that f_1, f_2 are also bounded, the product of f_1 and f_2 is Lipschitz too. More precisely, if A_1, A_2 are nonnegative real numbers such that

$$|f_1(x)| \le A_1, \quad |f_2(x)| \le A_2 \tag{2.12}$$

for all $x \in M$, then the product of f_1 and f_2 is $(A_1\, L_2 + A_2\, L_1)$-Lipschitz.

3 Connectedness

Let M be a nonempty set equipped with a distance function $d(x, y)$, and let ϵ be a positive real number. By an *ϵ-chain* in M we mean a finite sequence $w_0, \ldots, w_l$ of points in M such that

$$d(w_i, w_{i-1}) \le \epsilon \tag{3.1}$$

for $i = 1, \ldots, l$. A subset E of M is said to be *ϵ-connected* if for every pair of points $x, y \in E$ there is an ϵ-chain $w_0, \ldots, w_l$ of points in E such that $w_0 = x$ and $w_l = y$.

Thus a subset E of M is *not* ϵ-connected if there is a pair of points in E which cannot be connected by an ϵ-chain. This is equivalent to saying that E can be expressed as the union of two nonempty sets A, B such that

$$d(u, v) > \epsilon \tag{3.2}$$

for all $u \in A$ and $v \in B$. For in this case there is no ϵ-chain in E connecting points in A to points in B. Conversely, if E is not chain connected, so that there exist $x, y \in E$ which cannot be connected by an ϵ-chain of points in E, then one can define A to be the set of points in E which can be connected by an ϵ-chain of points in E to x, and B to be the set of remaining points in E, and A, B have the properties described earlier.

Suppose that M_1, M_2 are nonempty sets and that d_1, d_2 are distance functions on them. Let $\epsilon > 0$ be given, and assume that E is an ϵ-connected subset of M_1. Also let f be an L-Lipschitz mapping from M_1 to M_2. It is easy to see that $f(E)$ is then an $L\,\epsilon$-connected subset of M_2. There are analogous statements for more general "uniformly continuous" mappings, with more complicated adjustments to ϵ.

Let M be a nonempty set with distance function d and let E be a subset of M. We say that E is *chain-connected* if E is ϵ-connected for all $\epsilon > 0$. As in the preceding paragraph, the image of a chain-connected set under a uniformly continuous mapping is also chain connected.

A subset E of M is not chain connected if it is not ϵ-connected for some $\epsilon > 0$, which is equivalent to saying that there is an $\epsilon > 0$ such that E can be expressed as the union of two nonempty sets A, B, with $d(u, v) > \epsilon$ for all $u \in A$ and $v \in B$. In particular, a subset of a metric space which is not chain connected is not connected in the sense discussed in [15].

This is equivalent to saying that a subset of a metric space which is connected in the sense of [15] is also chain connected. The converse is not

true in general, however. For instance, the set of rational numbers, as a subset of the real line with the standard metric $|x - y|$, is chain connected but not connected.

Note that a basic result about connected subsets of metric spaces is that the image of a connected set under a continuous mapping is connected too. For a compact subset of a metric space, one can show that connectedness and chain connectedness are equivalent.

4 Hausdorff content

Let M be a nonempty set and $d(x, y)$ a metric on M, so that M becomes a metric space. A subset E of M is said to be *bounded* if the set of real numbers of the form $d(x, y)$ for $x, y \in E$ has an upper bound. The least upper bound or supremum of this set of numbers is called the diameter of E and is denoted $\operatorname{diam} E$, at least if E is not the empty set. We can also define the diameter of the empty set to be equal to 0. If E is a bounded subset of M, then the closure $\overline{E}$ of E, as discussed in [15], is also a bounded subset of M, and with the same diameter as E.

Let α be a positive real number. If $A_1, \ldots, A_k$ is a finite collection of bounded subsets of M, then consider the quantity

$$\sum_{i=1}^{k} (\operatorname{diam} A_i)^\alpha, \tag{4.1}$$

which is a kind of α-dimensional measurement of size of the A_i's. If E is a bounded subset of M, then we define $\mathcal{H}^\alpha(E)$, a version of the α-dimensional Hausdorff content of E, to be the infimum of the sums (4.1) over all finite collections $A_1, \ldots, A_k$ of bounded subsets of M such that

$$E \subseteq \bigcup_{i=1}^{k} A_i. \tag{4.2}$$

For instance, we could take $k = 1$ and $A_1 = E$, in which case (4.1) reduces to $(\operatorname{diam} E)^\alpha$. In particular we automatically have that

$$\mathcal{H}^\alpha(E) \le (\operatorname{diam} E)^\alpha. \tag{4.3}$$

As a special case we have that $\mathcal{H}^\alpha$ of the empty set is equal to 0. Also, if E_1, E_2 are two bounded subsets of M such that $E_1 \subseteq E_2$, then

$$\mathcal{H}^\alpha(E_1) \le \mathcal{H}^\alpha(E_2), \tag{4.4}$$

because any covering of E_2 by finitely many bounded subsets $A_1, \ldots, A_k$ of M is also a covering of E_1.

For any bounded subset E of M,

$$\mathcal{H}^\alpha(\overline{E}) = \mathcal{H}^\alpha(E). \tag{4.5}$$

Indeed, $\mathcal{H}^\alpha(E) \le \mathcal{H}^\alpha(\overline{E})$ simply because $E \subseteq \overline{E}$. Conversely, suppose that $A_1, \ldots, A_k$ are finitely many bounded subsets of M such that E is contained in the union of the A_i's. Then the closure of E is contained in the union of the closure of the A_i's, and since the diameter of the closure of a set is equal to the diameter of a set, it follows that $\mathcal{H}^\alpha(\overline{E}) \le \mathcal{H}^\alpha(E)$.

If E, F are bounded subsets of M, then

$$\mathcal{H}^\alpha(E \cup F) \le \mathcal{H}^\alpha(E) + \mathcal{H}^\alpha(F). \tag{4.6}$$

To see this, let $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ be arbitrary finite collections of bounded subsets of M such that

$$E \subseteq \bigcup_{i=1}^k A_i, \quad F \subseteq \bigcup_{j=1}^l B_j. \tag{4.7}$$

Then the combined collection $A_1, \ldots, A_k, B_1, \ldots, B_l$ is a finite collection of bounded subsets of M such that the union of E and F is contained in the union of the A_i's and B_j's together, and it follows that

$$\mathcal{H}^\alpha(E \cup F) \le \sum_{i=1}^k (\operatorname{diam} A_i)^\alpha + \sum_{j=1}^l (\operatorname{diam} B_j)^\alpha. \tag{4.8}$$

Because $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ are arbitrary finite collections of bounded subsets of M which cover E and F, respectively, we get (4.6), as desired.

Now suppose that M_1, M_2 are metric spaces with metrics d_1, d_2, respectively, and that f is an L-Lipschitz mapping from M_1 to M_2. If E is a bounded subset of M_1, then

$$\mathcal{H}^\alpha_{M_2}(f(E)) \le L^\alpha \, \mathcal{H}^\alpha_{M_1}(E), \tag{4.9}$$

where the subscripts M_1, M_2 for $\mathcal{H}^\alpha$ are used to indicate explicitly in which metric space one is working. Indeed, if A is any bounded subset of M_1, then $f(A)$ is a bounded subset of M_2, and the diameter of $f(A)$ in M_2 is less than or equal to L times the diameter of A in M_1. If $A_1, \ldots, A_k$ is a finite

collection of bounded subsets of M_1 such that E is contained in the union of the A_i's, then $f(A_1), \ldots, f(A_k)$ is a finite collection of bounded subsets in M_2 and $f(E)$ is contained in the union of the $f(A_i)$'s. The sum of the α powers of the diameters of the $f(A_i)$'s in M_2 is less than or equal to L^α times the sum of the α powers of the diameters of the A_i's in M_1, which leads to (4.9).

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